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Magnetic moment, vorticity-spin coupling and parity-odd conductivity of chiral fermions in 4-dimensional Wigner functions

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ABSTRACT

We demonstrate the emergence of the magnetic moment and spin-vorticity coupling of chiral fermions in 4-dimensional Wigner functions. In linear response theory with space–time varying electromagnetic fields, the parity-odd part of the electric conductivity can also be derived which reproduces results of the one-loop and the hard-thermal or hard-dense loop. All these properties show that the 4-dimensional Wigner functions capture comprehensive aspects of physics for chiral fermions in electromagnetic fields.

1. Introduction

Significant progress has been made in understanding the dynamics of chiral fermions in electromagnetic fields. This is particularly interesting in high energy heavy ion collisions where very strong magnetic fields can be created. The magnetic fields are so strong that quarks can be polarized and their momentum directions are parallel or anti-parallel to the magnetic field depending on quark chiralities and charges. Quarks with the same charge tend to move in the same direction. Any imbalance in the number of right-handed and left-handed quarks as a consequence of topological configurations of gauge fields may lead to such a charge separation effect which can be tested in experiments [1]. This is termed as the Chiral Magnetic Effect (CME) [2,3]. The Chiral Vortical Effect (CVE) is also an accompanying effect due to rotation in a relativistic and charged fluid [4,5]. The interplay of chiral magnetic and chiral separation effects may lead to a phenomenon called the Chiral Magnetic Wave [6], whose vortical counter part is the Chiral Vortical Wave [7].

Kinetic theory is an important tool to describe these phenomena in phase space of chiral fermions. The Abelian Berry potential takes an important role in 3-dimensions (3D) kinetic approach in accommodation of axial anomaly [8–10]. It has been shown that the CME, CVE and Covariant Chiral Kinetic Equation (CCKE) can be derived in quantum kinetic theory from the Wigner function in 4-dimensions (4D) in external electromagnetic fields [11,12]. The 3D chiral kinetic equation [8–10] can be obtained from the CCKE by integrating out the zero component of the 4-momentum.

In the 3D chiral kinetic equation, it has been shown that the fermion energy is shifted by the interaction energy of magnetic moment with the magnetic field [13]. The magnetic moment and spin of fermions have relativistic origin [14–16]. It is a natural conjecture that the magnetic moment should also emerge in the covariant quantum kinetic approach in 4D Wigner functions. In this paper, we will demonstrate the emergence of the magnetic moment as well as spin-vorticity coupling in the framework of covariant quantum kinetic theory based on 4D Wigner functions. We will also show that the parity-odd part of electric conductivity (chiral magnetic conductivity) can also be derived from 4D Wigner functions in linear response theory with space–time varying electromagnetic fields. The result reproduces the chiral magnetic conductivity of one loop [17] and hard-thermal or hard-dense loop (HTL or HDL) [18,19] under proper approximations [13,20].

We adopt the same sign conventions for the fermion charge Q and the axial vector component of the Wigner function as in Refs. [11,12,21].

2. Wigner functions for chiral fermions

In a background electromagnetic field, the quantum mechanical anologue of a classical phase-space distribution for fermions is the gauge invariant Wigner function $W_{\alpha\beta}(x,p)$ which satisfies the equation of motion [21,22], $\left(\gamma_\mu K^\mu - m\right) W(x,p) = 0$, where

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$x=(x_0,\mathbf{x})$ and $p=(p_0,\mathbf{p})$ are space–time and energy–momentum 4-vectors. For the constant field strength $F_{\mu\nu}$, the operator K^μ is defined by $K^\mu = p^\mu + i\frac{1}{2}\nabla^\mu$ with $\nabla^\mu = \partial_x^\mu - QF^{\mu\nu}\partial_\nu^p$. The Wigner function can be decomposed into 16 independent generators of Clifford algebra, whose coefficients $\mathscr{F}$, $\mathscr{P}$, $\mathscr{V}_\mu$, $\mathscr{A}_\mu$ and $\mathscr{S}_{\mu\nu}$ are the scalar, pseudo-scalar, vector, axial-vector and tensor components of the Wigner function respectively. For massless or chiral fermions, the equations for $\mathscr{V}_\mu$ and $\mathscr{A}_\mu$ are decoupled from other components of the Wigner function.

3. Formal solution to quantum kinetic equations

For chiral (massless) fermions, we can rewrite the equations for $\mathscr{V}_\mu$ and $\mathscr{A}_\mu$ into those for right-handed ($s=+$) and left-handed ($s=-$) vectors $\mathscr{J}_\mu^s(x,p)$,

$$\begin{aligned} p^\mu \mathscr{J}_\mu^s(x,p) &= 0, \\ \nabla^\mu \mathscr{J}_\mu^s(x,p) &= 0, \\ 2s(p^\lambda \mathscr{J}_s^\rho - p^\rho \mathscr{J}_s^\lambda) &= -\epsilon^{\mu\nu\lambda\rho}\nabla_\mu \mathscr{J}_\nu^s, \end{aligned} \tag{1}$$

where $\mathscr{J}_\mu^s(x,p)$ are given by

$$\mathscr{J}_\mu^s(x,p) = \frac{1}{2}[\mathscr{V}_\mu(x,p) + s\mathscr{A}_\mu(x,p)]. \tag{2}$$

One can derive a formal solution of $\mathscr{J}_\mu^s$ in Eq. (1) by a perturbation method in powers of ∇^μ and $F_{\mu\nu}$. The solutions at the zeroth and first order were given by Refs. [11,12] and can be cast into the following form,

$$\begin{aligned} \mathscr{J}_{(0)s}^\rho(x,p) &= p^\rho f_s \delta(p^2), \\ \mathscr{J}_{(1)s}^\rho(x,p) &= -\frac{s}{2}\tilde{\Omega}^{\rho\beta}p_\beta \frac{df_s}{dp_0}\delta(p^2) - \frac{s}{p^2}Q\tilde{F}^{\rho\lambda}p_\lambda f_s \delta(p^2), \end{aligned} \tag{3}$$

where $p_0 \equiv u\cdot p$ with u^μ being the fluid velocity, and f_s are distribution functions of chiral fermions defined by

$$f_s(x,p) = \frac{2}{(2\pi)^3}\left[\Theta(p_0)f_F(p_0-\mu_s) + \Theta(-p_0)f_F(-p_0+\mu_s)\right]. \tag{4}$$

Here $f_F(y) \equiv 1/[\exp(\beta y)+1]$ is the Fermi–Dirac distribution function, and $\beta = 1/T$ and μ_s are the temperature inverse and the chemical potential of chiral fermions with chirality s respectively. We can decompose μ_s into the scalar and pseudo-scalar parts, $\mu_s = \mu + s\mu_5$. We have used the following formula, $\tilde{F}^{\rho\lambda} = \frac{1}{2}\epsilon^{\rho\lambda\mu\nu}F_{\mu\nu}$, $\tilde{\Omega}^{\xi\eta} = \frac{1}{2}\epsilon^{\xi\eta\nu\sigma}\Omega_{\nu\sigma}$ and $\Omega_{\nu\sigma} = \frac{1}{2}(\partial_\nu u_\sigma - \partial_\sigma u_\nu)$, where $\epsilon^{\mu\nu\sigma\beta}$ and $\epsilon_{\mu\nu\sigma\beta}$ are anti-symmetric tensors with $\epsilon^{\mu\nu\sigma\beta} = 1(-1)$ and $\epsilon_{\mu\nu\sigma\beta} = -1(1)$ for even (odd) permutations of indices 0123, so we have $\epsilon^{0123} = -\epsilon_{0123} = 1$. Instead of $\Omega_{\nu\sigma}$, $\tilde{\Omega}^{\xi\eta}$, $F_{\mu\nu}$ and $\tilde{F}^{\rho\lambda}$, we will also use the vorticity vector $\omega^\rho = \frac{1}{2}\epsilon^{\rho\sigma\alpha\beta}u_\sigma\partial_\alpha u_\beta$, the electric field $E^\mu = F^{\mu\nu}u_\nu$, and the magnetic field $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}u_\nu F_{\lambda\rho}$.

4. Magnetic moment and energy shift

In this section we will derive the magnetic moment and energy shift from the formal solution of the Wigner function (3). To this end we assume that the chemical potential μ_s depends on space–time and 3-momentum. For convenience, we will focus on the electromagnetic term in this section and turn off the vortical term. The vortical term will be investigated separately in the next section.

If μ_s does not depend on the 3-momentum, from Eq. (3), we obtain the fermion number current j_s^ρ (the electric current should be Qj_s^ρ),

$$j_s^\rho(x) = \int d^4p\, \mathscr{J}_s^\rho(x,p) = n_s^{(0)}u^\rho + \sigma_{\chi,s}^{(0)}B^\rho, \tag{5}$$

where $\sigma_{\chi,s}^{(0)} = sQ\frac{\mu_s}{4\pi^2}$ is the parity-odd part of the conductivity in the static limit for chiral fermions with chirality s. Since $f_s(x,p)$ does not depend on 3-momentum, the momentum integral involving $\mathscr{J}_{(0)s}^\rho$ is non-vanishing only along the fluid velocity u^ρ, so one can just make replacement $p^\rho \to p_0 u^\rho$. Here $n_s^{(0)}$ is the fermion number density of a non-interacting gas of chiral fermions. For simplicity of notation, we will work in the local rest frame of a fluid cell with $u^\mu = (1,0,0,0)$, then we have $E_p = |\mathbf{p}|$. We will use the velocity on-shell vector $v^\rho = (1,\mathbf{v})$ with the 3-velocity $\mathbf{v} = \mathbf{p}/E_p$.

In order to extract the magnetic moment and its energy shift, we can extend our previous scenario by assuming that μ_s depends on the 3-momentum since the energy shift from the magnetic moment can be grouped into the effective chemical potential, while the magnetic moment of a chiral fermion is proportional to its spin which is polarized along its momentum. We can write the chemical potential into the sum of a constant part and a $(x,\mathbf{p})$-dependent part, $\mu_s^e(x,\mathbf{p}) \approx \mu_s + W_s^e(x,\mathbf{p})$, where μ_s do not depend on x and $\mathbf{p}$, and $e=\pm 1$ denote positive/negative energy corresponding to fermions ($e=+$) and anti-fermions ($e=-$). The quantities $W_s^e(x,\mathbf{p})$ can be further decomposed as $W_s^e(x,\mathbf{p}) = W(x,\mathbf{v}) + \frac{se}{2E_p}W_5(x,\mathbf{v})$, where the introduction of the factor $1/(2E_p)$ in the second term is to make W_5 depends on $\mathbf{v}$ only. Note that our definitions of μ_5, W and W_5 can be matched to those in Ref. [20] up to constants. In this Letter, we will not use the specific form of $\mu_s^e(x,\mathbf{p})$ except its formal dependence on e, s, x and $\mathbf{p}$.

With $\mu_s^e(x,\mathbf{p})$, $f_s(x,p)$ in Eq. (4) now becomes

$$f_s(x,p) = \frac{2}{(2\pi)^3}\sum_{e=\pm 1}\Theta(ep_0)f_F[ep_0 - e\mu_s^e(x,e\mathbf{p})]. \tag{6}$$

Note that there is a minus sign ($e=-1$) in front of $\mathbf{p}$ in the anti-fermion sector. So the fermion number 4-current $j_s^\rho(x)$ can be obtained by integrating $\mathscr{J}_s^\rho(x,p)$ in Eq. (3) over 4-momentum with f_s given by Eq. (6). The fermion number density can be derived in Appendix A and be put into a compact form

$$\begin{aligned} n_s &= \int \frac{d^3p}{(2\pi)^3}\sum_{e=\pm 1} ef_F[E_p - e\mu_s^e(x,\mathbf{p})] \\ &\quad + \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p^2}sQ(\mathbf{v}\cdot\mathbf{B})\sum_{e=\pm 1} ef_F[E_p - e\mu_s^e(x,\mathbf{p})] \\ &\quad - \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p^2}E_p sQ(\mathbf{v}\cdot\mathbf{B})\frac{d}{dE_p}\sum_{e=\pm 1} ef_F[E_p - e\mu_s^e(x,\mathbf{p})] \\ &\approx \int \frac{d^3p}{(2\pi)^3}\sqrt{\gamma}\sum_{e=\pm 1} ef_F[E'_p - e\mu_s^e(x,\mathbf{p})], \end{aligned} \tag{7}$$

where $\sqrt{\gamma} \equiv (1 + sQ\mathbf{a}_p\cdot\mathbf{B})$ is the phase-space measure with the Berry curvature $\mathbf{a}_p = \frac{1}{2E_p^2}\mathbf{v}$, and $E'_p = E_p(1 - sQ\mathbf{a}_p\cdot\mathbf{B})$ is the effective energy. So from E'_p we can read out the energy shift and magnetic moment of chiral fermions,

$$\Delta E_B = -\boldsymbol{\mu}_m\cdot\mathbf{B} \tag{8}$$

$$\boldsymbol{\mu}_m = g\frac{eQ}{2|\mathbf{p}|}\mathbf{S} = \frac{sQ}{2|\mathbf{p}|}\mathbf{v} \tag{9}$$

where $\mathbf{S} = \frac{1}{2}es\mathbf{v}$ and $g = 2$ are the spin and g-factor of charged chiral fermions respectively. For $e = s = +$ or $e = s = -$ (positive energy and right-hand chirality or negative energy and left-hand chirality), $\mathbf{S} = \frac{1}{2}\mathbf{v}$ (positive helicity), otherwise $\mathbf{S} = -\frac{1}{2}\mathbf{v}$ (negative helicity). However $\boldsymbol{\mu}_m$ only depends on chirality and charge of positive energy fermions. For $s = Q = +$ or $s = Q = -$ (positive chirality and positive charge or negative chirality and negative charge), $\boldsymbol{\mu}_m$ is parallel to $\mathbf{p}$, otherwise it is anti-parallel to $\mathbf{p}$. We also see the emergence of the phase-space measure $\sqrt{\gamma}$, which is not surprising since we have already shown this in Ref. [12]. What is new is the emergence of the magnetic moment and energy shift in our Lorentz covariant kinetic approach although superficially there is no such terms in the distribution function f_s in Eq. (6). We then reproduce the results of Son and Yamamoto [13,20,23].

5. Energy shift from spin-vorticity coupling

We now consider the vorticity term in the Wigner function $\mathscr{J}^{\rho}_{(1)s}$ in Eq. (3). The fermion number current from vorticity term is given by

$$\begin{aligned} j^{\rho}_{(1)s}&(x)\\ &= -\frac{s}{2}\int d^4p\,\tilde{\Omega}^{\rho\beta}p_\beta \frac{df_s}{dp_0}\delta(p^2)\\ &= s\frac{1}{4\pi^2}\left(\frac{\pi^2}{3}T^2+\mu_s^2\right)\omega^\rho\\ &\quad + u^\rho\int \frac{d^3p}{(2\pi)^3}\sum_{e=\pm1}(-\frac{es}{2}\boldsymbol{\omega}\cdot\mathbf{v})e\frac{d}{dE_p}f_F[E_p - e\mu_s^e(x,\mathbf{p})], \end{aligned} \tag{10}$$

where the first term ($\sim \omega^\rho$) gives the CVE coefficient, while the second term $\sim u^\rho$ provides the correction to the fermion number density from the spin-vorticity coupling energy. Note that these two terms are orthogonal to each other since $u\cdot\omega = 0$, so there is no correction to the CVE coefficient at this level from the second term. We note that there are a few arguments about the temperature dependent part of the CVE coefficient such as its origin and whether there are corrections from high order terms [24–26]. The u^ρ part of $j^{\rho}_{s(1)}$ in Eq. (10) gives the vorticity contribution to n_s in Eq. (7). So the spin-vorticity energy shift can be read out from Eq. (10),

$$\Delta E_\omega = -\frac{1}{2}es(\boldsymbol{\omega}\cdot\mathbf{v}) = -\boldsymbol{\omega}\cdot\mathbf{S}, \tag{11}$$

where $\boldsymbol{\omega}$ is the vorticity 3-vector. From the energy density one can also derive the correction from the vorticity term of $\mathscr{J}^{\rho}_{(1)s}$,

$$\epsilon_{s(1)} = \int \frac{d^3p}{(2\pi)^3}E_p\sum_{e=\pm1}(-\frac{es}{2}\boldsymbol{\omega}\cdot\mathbf{v})\frac{d}{dE_p}f_F[E_p - e\mu_s^e(x,\mathbf{p})]. \tag{12}$$

One can also read out the same energy shift ΔE_ω from the spin-vorticity coupling as (11).

Combining Eq. (8) and (11), we obtain the total effective energy with energy shifts from the magnetic moment and the spin-vorticity coupling,

$$E'_p = E_p - \boldsymbol{\mu}_m\cdot\mathbf{B} - \boldsymbol{\omega}\cdot\mathbf{S}. \tag{13}$$

We see the emergence of the magnetic moment and the spin-vorticity coupling from the Wigner function solution (3).

6. Parity-odd electric conductivity

We can calculate the parity-odd part of the electric conductivity or chiral magnetic conductivity, $\sigma_\chi(\omega,\mathbf{k})$, from the Wigner function solution (3) as the result of expansion in $\nabla^\mu = \partial_x^\mu - QF^{\mu\nu}\partial_\nu^p$, where we assume that $\partial_x^\mu \sim (\omega,\mathbf{k})$ is of the same order as $F^{\mu\nu}$. This is valid for small ω and $|\mathbf{k}|$. For finite values of ω and $|\mathbf{k}|$, a more rigorous way is to carry out the expansion of linear response in space–time varying fields only and regard $\partial_x^\mu \sim (\omega,\mathbf{k})$ as $O(1)$ quantities. In this case, one should solve a different set of equations with similar structure to Eq. (1) but with replacements $p^\mu \to p^\mu - \frac{1}{2}Qj_1\left(\frac{1}{2}\Delta\right)F^{\mu\nu}\partial_\nu^p$ and $\nabla^\mu \to \partial_x^\mu - Qj_0\left(\frac{1}{2}\Delta\right)F^{\mu\nu}\partial_\nu^p$, where $\Delta \equiv \partial_x\cdot\partial_p$ and ∂_x acts only on $F^{\mu\nu}$ but not on distributions. Here $j_0(y) = \sin(y)/y$ and $j_1(y) = (\sin y - y\cos y)/y^2$ are spherical Bessel functions. The solution to the leading equations is still $\mathscr{J}^{\rho}_{(0)s} = p^\rho f_s(p)\delta(p^2)$, same as in Eqs. (3), (4) except that $f_s(p)$ does not depend on x because $\mathscr{J}^{\rho}_{(0)s}$ should satisfy $\partial^x_\mu\mathscr{J}^{\mu}_{s(0)} = 0$. We refer the readers to Appendix B for the details of Wigner equations and its solutions in this case.

As shown in Appendix B, the parity-odd part of the Wigner functions in $k = (\omega,\mathbf{k})$ representation which linearly depends on fields can be written in the following form

$$\begin{aligned} \mathscr{J}^{s}_{(1)\mu}&(k,p)\\ &= -i\frac{sQ}{2p\cdot k}\epsilon_{\mu\nu\rho\sigma}k^\nu p^\sigma A^\rho(k)j_0\left(\frac{1}{2}\Delta\right)(k\cdot\partial_p)[f_s\delta(p^2)]\\ &= i\frac{sQ}{2p\cdot k}\epsilon_{\mu\nu\rho\sigma}k^\nu p^\rho A^\sigma\left\{f_s\left(p+\frac{1}{2}k\right)\delta\left[\left(p+\frac{1}{2}k\right)^2\right]\right.\\ &\quad\left. - f_s\left(p-\frac{1}{2}k\right)\delta\left[\left(p-\frac{1}{2}k\right)^2\right]\right\}, \end{aligned} \tag{14}$$

where we have used $\Delta = -ik\cdot\partial_p$, $j_0\left(\frac{1}{2}\Delta\right)(k\cdot\partial_p) = 2i\sin\left(\frac{1}{2}\Delta\right)$ and $\exp\left(\frac{1}{2}k\cdot\partial_p\right)f_s\delta(p^2) = f_s\left(p+\frac{1}{2}k\right)\delta\left[\left(p+\frac{1}{2}k\right)^2\right]$. We can obtain the 3-vector current by integration over 4-momentum for the spatial component of $\mathscr{J}^{s}_{(1)\mu}$, i.e. $\mathbf{j}^i_s(\omega,\mathbf{k}) = -\int d^4p\,\mathscr{J}^{s}_{(1),\mu=i}$. We can then read out $\sigma^s_\chi(\omega,\mathbf{k})$ which is just the one-loop result, Eq. (36) of Ref. [17],

$$\begin{aligned} \sigma^s_\chi(\omega,\mathbf{k}) = \frac{sQ}{16\pi^2}\frac{\mathbf{k}^2-\omega^2}{|\mathbf{k}|^3}&\int d|\mathbf{p}|\,[f_F(|\mathbf{p}|-\mu_s) - f_F(|\mathbf{p}|+\mu_s)]\\ &\left[(2|\mathbf{p}|-\omega)\ln\frac{(\omega-|\mathbf{p}|)^2-(|\mathbf{p}|+|\mathbf{k}|)^2}{(\omega-|\mathbf{p}|)^2-(|\mathbf{p}|-|\mathbf{k}|)^2}\right.\\ &\left.+(2|\mathbf{p}|+\omega)\ln\frac{(\omega+|\mathbf{p}|)^2-(|\mathbf{p}|+|\mathbf{k}|)^2}{(\omega+|\mathbf{p}|)^2-(|\mathbf{p}|-|\mathbf{k}|)^2}\right]. \end{aligned} \tag{15}$$

If we assume that external frequency and momentum are much smaller than internal momentum, $\omega, |\mathbf{k}| \ll |\mathbf{p}|$, we can reproduce the HTL or HDL result [13,20],

$$\sigma_\chi^{\mathrm{HTL/HDL}}(\omega,\mathbf{k}) = \sigma_\chi^{(0)}\left(1-\frac{\omega^2}{|\mathbf{k}|^2}\right)\left[1-\frac{\omega}{2|\mathbf{k}|}\ln\frac{\omega+|\mathbf{k}|}{\omega-|\mathbf{k}|}\right], \tag{16}$$

where we have used $\sigma_\chi^{(0)} = \sigma_{\chi,\mathrm{R}}^{(0)} + \sigma_{\chi,\mathrm{L}}^{(0)} = \frac{1}{2\pi^2}Q\mu_5$. By including finite damping rate into fermionic propagators the non-commutativity of the static limit $\omega \to 0$ and $|\mathbf{k}| \to 0$ can be removed [23].

7. Summary

We have investigated properties of chiral fermions in electromagnetic fields in covariant quantum kinetic theory based on 4D Wigner functions. We have shown that the energy shifts of chiral fermions from the magnetic moment and the spin-vorticity coupling emerge from the Wigner function solutions derived from our previous works. We have also calculated the parity-odd electric conductivity in linear response theory and reproduced the results of one-loop and HTL/HDL. All these properties, together with previous findings, show that the 4D Wigner functions capture comprehensive aspects of physics for chiral fermions in electromagnetic fields.

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Appendix A. Derivation of Eq. (7)

In the case with momentum dependent chemical potential $\mu_s^e(x,\mathbf{p})$, the zeroth order term of j_s^ρ which corresponds to $\mathscr{J}^\rho_{(0)s}$ in Eq. (3) can be derived as

$$\begin{aligned}
&j^\rho_{(0)s}(x)\\
&\quad=\int d^4p f_s(x,p)p^\rho\delta(p^2)\\
&\quad=\int\frac{d^3p}{(2\pi)^3}\left[(u^\rho+\tilde v^\rho)f_F(E_p-\mu_s^+(x,\mathbf{p}))\right.\\
&\qquad\left.+(-u^\rho+\tilde v^\rho)f_F(E_p+\mu_s^-(x,-\mathbf{p}))\right]\\
&\quad=\int\frac{d^3p}{(2\pi)^3}v^\rho\left[f_F(E_p-\mu_s^+(x,\mathbf{p}))-f_F(E_p+\mu_s^-(x,\mathbf{p}))\right].
\end{aligned}\tag{A.1}$$

In the above equation, we have substituted $f_s(x,p)$ of Eq. (6), used $\delta(p^2)=\frac{1}{2E_p}[\delta(p_0-E_p)+\delta(p_0+E_p)]$, and carried out the integral over p_0. We have used the notation $\tilde v^\rho\equiv(0,\mathbf{v})$ with $\mathbf{v}\equiv\mathbf{p}/E_p=\mathbf{p}/|\mathbf{p}|$. From the second to the third equality, we have changed the integral variable $\mathbf{p}\to-\mathbf{p}$ for the second term (the negative energy sector) inside the square brackets.

The first order term of j_s^ρ corresponding to the electromagnetic part of $\mathscr{J}^\rho_{(1)s}$ in Eq. (3) can be written as

$$\begin{aligned}
j^\rho_{(1)s}(x)&=sQ\,\tilde F^{\rho\lambda}\int d^4p f_s(x,p)p_\lambda\delta'(p^2)\\
&=\frac{s}{2}Q\,\tilde F^{\rho\lambda}\int d^4p f_s(x,p)\frac{p_\lambda}{p_0}\frac{d}{dp_0}\delta(p^2)\\
&=-\frac{s}{2}Q\,\tilde F^{\rho\lambda}\int d^4p\frac{d}{dp_0}\left[f_s(x,p)\frac{p_\lambda}{p_0}\right]\delta(p^2)\\
&\quad+\frac{s}{2}Q\,\tilde F^{\rho\lambda}\int d^4p\frac{d}{dp_0}\left[f_s(x,p)\frac{p_\lambda}{p_0}\delta(p^2)\right]\\
&=-\frac{s}{2}Q\,\tilde F^{\rho\lambda}\int d^4p\left[\frac{p_\lambda}{p_0}\cdot\frac{df_s(x,p)}{dp_0}-f_s(x,p)\frac{\tilde p_\lambda}{p_0^2}\right]\delta(p^2).
\end{aligned}\tag{A.2}$$

In the first equality we have used the identities $\delta'(p^2)\equiv\frac{d}{dp^2}\delta(p^2)=-\frac{1}{p^2}\delta(p^2)$ and $\frac{d}{dp^2}\delta(p^2)=\frac{d}{dp_0^2}\delta(p^2)$. From the third to the fourth equality, we have dropped the integral of complete derivative since $f_s(x,p)\frac{p_\lambda}{p_0}\delta(p^2)\to0$ at the boundaries $p_0\to\pm\infty$. In the last equality we have used the notation $\tilde p^\lambda=(0,\mathbf{p})$. In order to further simplify Eq. (A.2), we will use

$$\begin{aligned}
\frac{df_s(x,p)}{dp_0}=\frac{2}{(2\pi)^3}&\left[\Theta(p_0)\frac{d}{dp_0}f_F(p_0-\mu_s^+(x,\mathbf{p}))\right.\\
&\left.-\Theta(-p_0)\frac{d}{d(-p_0)}f_F(-p_0+\mu_s^-(x,-\mathbf{p}))\right]
\end{aligned}\tag{A.3}$$

where we have neglected derivatives of $\Theta(\pm p_0)$ because they vanish when combining with $\delta(p^2)$. Inserting Eqs. (6), (A.3) into the last equality of Eq. (A.2), we obtain

$$\begin{aligned}
j^\rho_{(1)s}(x)=&-\frac{s}{2}Q\,\tilde F^{\rho\lambda}\int\frac{d^3p}{(2\pi)^3E_p}\left[(u_\lambda+\tilde v_\lambda)\frac{df_F(E_p-\mu_s^+(x,\mathbf{p}))}{dE_p}\right.\\
&\left.+(-u_\lambda+\tilde v_\lambda)\frac{df_F(E_p+\mu_s^-(x,-\mathbf{p}))}{dE_p}\right]\\
&+\frac{s}{2}Q\,\tilde F^{\rho\lambda}\int\frac{d^3p}{(2\pi)^3E_p^2}\tilde v_\lambda\left[f_F(E_p-\mu_s^+(x,\mathbf{p}))\right.\\
&\left.+f_F(E_p+\mu_s^-(x,-\mathbf{p}))\right]\\
=&-\frac{s}{2}Q\int\frac{d^3p}{(2\pi)^3E_p}\tilde F^{\rho\lambda}v_\lambda\left[\frac{df_F(E_p-\mu_s^+(x,\mathbf{p}))}{dE_p}\right.\\
&\left.-\frac{df_F(E_p+\mu_s^-(x,\mathbf{p}))}{dE_p}\right]\\
&+\frac{s}{2}Q\int\frac{d^3p}{(2\pi)^3E_p^2}\tilde F^{\rho\lambda}\tilde v_\lambda\left[f_F(E_p-\mu_s^+(x,\mathbf{p}))\right.\\
&\left.-f_F(E_p+\mu_s^-(x,\mathbf{p}))\right],
\end{aligned}\tag{A.4}$$

where we have changed in the second equality the integral variable $\mathbf{p}\to-\mathbf{p}$ for the second terms (sectors of negative energy) inside square brackets, and we have also used $v_\lambda=u_\lambda+\tilde v_\lambda$.

From Eqs. (A.1), (A.4), we can obtain the fermion number density by contraction of the fluid velocity u_ρ and $j_s^\rho=j^\rho_{(0)s}+j^\rho_{(1)s}$, $n_s=u_\rho j_s^\rho(x)$, which gives Eq. (7) by using $u_\rho\tilde v^\rho=0$ and $u_\rho\tilde F^{\rho\lambda}v_\lambda=u_\rho\tilde F^{\rho\lambda}\tilde v_\lambda=\mathbf{B}\cdot\mathbf{v}$.

Appendix B. Wigner functions in space–time varying electromagnetic fields

We have discussed in Sections 2 and 3 that Eq. (1) holds for the constant field strength $F_{\mu\nu}$. For space–time varying $F_{\mu\nu}$, it takes the general form (see Eqs. (5.12–5.21) of Ref. [21]),

$$\begin{aligned}
&\Pi^\mu\mathscr{J}^s_\mu=0,\\
&G^\mu\mathscr{J}^s_\mu=0,\\
&2s(\Pi^\mu\mathscr{J}^\nu_s-\Pi^\nu\mathscr{J}^\mu_s)=-\epsilon^{\mu\nu\rho\sigma}G_\rho\mathscr{J}^s_\sigma.
\end{aligned}\tag{B.1}$$

The operators Π^μ and G^μ now replace p^μ and ∇^μ in Eq. (1) respectively,

$$\Pi^\mu = p^\mu - \frac{1}{2} j_1\left(\frac{1}{2}\Delta\right) Q F^{\mu\nu}\partial_\nu^p,$$
$$G^\mu = \partial_x^\mu - j_0\left(\frac{1}{2}\Delta\right) Q F^{\mu\nu}\partial_\nu^p. \quad \text{(B.2)}$$

Note that ∂_x acts only on $F^{\mu\nu}$ but not on other functions to its right. For constant $F_{\mu\nu}$, we recover $\Pi^\mu = p^\mu$ and $G^\mu = \nabla^\mu$ by using $j_0(0) = 1$ and $j_1(0) = 0$.

The solution to Eq. (B.1) can be found by perturbation in fields. To the first order, we can formally write

$$\mathscr{J}_s^\mu = \mathscr{J}_{(0)s}^\mu + \mathscr{J}_{(1)s}^\mu. \quad \text{(B.3)}$$

The zeroth order equations read

$$p^\mu \mathscr{J}_{(0)\mu}^s = 0,$$
$$\partial_x^\mu \mathscr{J}_{(0)\mu}^s = 0,$$
$$2s[p^\mu \mathscr{J}_{(0)s}^\nu - p^\nu \mathscr{J}_{(0)s}^\mu] = -\epsilon^{\mu\nu\rho\sigma}\partial_\rho^x \mathscr{J}_{(0)\sigma}^s, \quad \text{(B.4)}$$

whose solution is in the same form as the first line of Eq. (3) except that f_s does not depend on x here due to $\partial_x^\mu \mathscr{J}_{(0)\mu}^s = 0$. The first order equations read,

$$p^\mu \mathscr{J}_{(1)\mu}^s - \frac{1}{2} j_1\left(\frac{1}{2}\Delta\right) Q F^{\mu\nu}\partial_\nu^p \mathscr{J}_{(0)\mu}^s = 0,$$
$$\partial_x^\mu \mathscr{J}_{(1)\mu}^s - j_0\left(\frac{1}{2}\Delta\right) Q F^{\mu\nu}\partial_\nu^p \mathscr{J}_{(0)\mu}^s = 0,$$
$$-\epsilon_{\mu\nu\rho\sigma}\left[\partial_x^\rho \mathscr{J}_{(1)s}^\sigma - j_0\left(\frac{1}{2}\Delta\right) Q F^{\rho\lambda}\partial_\lambda^p \mathscr{J}_{(0)s}^\sigma\right]$$
$$= 2s[p_\mu \mathscr{J}_{(1)\nu}^s - p_\nu \mathscr{J}_{(1)\mu}^s]$$
$$+ s\left[j_1\left(\frac{1}{2}\Delta\right) Q F_{\nu\sigma}\partial_p^\sigma \mathscr{J}_{(0)\mu}^s - j_1\left(\frac{1}{2}\Delta\right) Q F_{\mu\sigma}\partial_p^\sigma \mathscr{J}_{(0)\nu}^s\right]. \quad \text{(B.5)}$$

Contracting ∂_x^ν with the third equation and using the second equation of Eq. (B.5), we arrive at

$$p\cdot\partial_x \mathscr{J}_{(1)\mu}^s(x,p)$$
$$= p_\mu j_0\left(\frac{1}{2}\Delta\right) Q F^{\rho\sigma}\partial_\sigma^p \mathscr{J}_{(0)\rho}^s$$
$$- \frac{1}{2} s\epsilon_{\mu\nu\rho\sigma}\partial_x^\nu\left[j_0\left(\frac{1}{2}\Delta\right) Q F^{\rho\lambda}\partial_\lambda^p \mathscr{J}_{(0)s}^\sigma\right]$$
$$+ \frac{1}{2} Q\partial_x^\nu\left[j_1\left(\frac{1}{2}\Delta\right)\left(F_{\nu\sigma}\partial_p^\sigma \mathscr{J}_{(0)\mu}^s - F_{\mu\sigma}\partial_p^\sigma \mathscr{J}_{(0)\nu}^s\right)\right]. \quad \text{(B.6)}$$

We can solve $\mathscr{J}_{(1)\mu}^s$ in momentum space by replacing $\partial_x \to -ik$ and $\Delta \to -ik\cdot\partial_p$. After a lengthy but straightforward calculation, we obtain

$$\mathscr{J}_{(1)\mu}^s(k,p)$$
$$= \frac{Q}{p\cdot k} p_\mu[(p\cdot k)(A\cdot\partial_p) - (p\cdot A)(k\cdot\partial_p)] j_0\left(\frac{1}{2}\Delta\right)[f_s\delta(p^2)]$$
$$- i\frac{sQ}{2p\cdot k}\epsilon_{\mu\nu\rho\sigma}k^\nu p^\sigma A^\rho j_0\left(\frac{1}{2}\Delta\right)(k\cdot\partial_p)[f_s\delta(p^2)]$$
$$+ \frac{Q}{4p\cdot k}[k_\mu(k\cdot A) - k^2 A_\mu](k\cdot\partial_p) j_0\left(\frac{1}{2}\Delta\right)[f_s\delta(p^2)]$$
$$+ i\frac{1}{2} Q[k_\mu(A\cdot\partial_p) - A_\mu(k\cdot\partial_p)] j_1\left(\frac{1}{2}\Delta\right)[f_s\delta(p^2)]. \quad \text{(B.7)}$$

The second term is the parity-odd part and can be put into the form of Eq. (14).

References

[1] B. Abelev, et al., STAR Collaboration, Phys. Rev. Lett. 103 (2009) 251601, arXiv:0909.1739.
[2] D.E. Kharzeev, L.D. McLerran, H.J. Warringa, Nucl. Phys. A 803 (2008) 227, arXiv:0711.0950.
[3] K. Fukushima, D.E. Kharzeev, H.J. Warringa, Phys. Rev. D 78 (2008) 074033, arXiv:0808.3382.
[4] D.T. Son, P. Surowka, Phys. Rev. Lett. 103 (2009) 191601, arXiv:0906.5044.
[5] D.E. Kharzeev, D.T. Son, Phys. Rev. Lett. 106 (2011) 062301, arXiv:1010.0038.
[6] Y. Burnier, D.E. Kharzeev, J. Liao, H.-U. Yee, Phys. Rev. Lett. 107 (2011) 052303, arXiv:1103.1307.
[7] Y. Jiang, X.-G. Huang, J. Liao, arXiv:1504.03201, 2015.
[8] D.T. Son, N. Yamamoto, Phys. Rev. Lett. 109 (2012) 181602, arXiv:1203.2697.
[9] M. Stephanov, Y. Yin, Phys. Rev. Lett. 109 (2012) 162001, arXiv:1207.0747.
[10] J.-W. Chen, J.-y. Pang, S. Pu, Q. Wang, Phys. Rev. D 89 (2014) 094003, arXiv:1312.2032.
[11] J.-H. Gao, Z.-T. Liang, S. Pu, Q. Wang, X.-N. Wang, Phys. Rev. Lett. 109 (2012) 232301, arXiv:1203.0725.
[12] J.-W. Chen, S. Pu, Q. Wang, X.-N. Wang, Phys. Rev. Lett. 110 (2013) 262301, arXiv:1210.8312.
[13] D.T. Son, N. Yamamoto, Phys. Rev. D 87 (2013) 085016, arXiv:1210.8158.
[14] J.-Y. Chen, D.T. Son, M.A. Stephanov, H.-U. Yee, Y. Yin, Phys. Rev. Lett. 113 (2014) 182302, arXiv:1404.5963.
[15] C. Duval, P. Horvathy, arXiv:1406.0718, 2014.
[16] C. Manuel, J.M. Torres-Rincon, Phys. Rev. D 90 (2014) 076007, arXiv:1404.6409.
[17] D.E. Kharzeev, H.J. Warringa, Phys. Rev. D 80 (2009) 034028, arXiv:0907.5007.
[18] E. Braaten, R.D. Pisarski, Nucl. Phys. B 337 (1990) 569.
[19] M. Laine, J. High Energy Phys. 0510 (2005) 056, arXiv:hep-ph/0508195.
[20] C. Manuel, J.M. Torres-Rincon, Phys. Rev. D 89 (2013) 096002, arXiv:1312.1158.
[21] D. Vasak, M. Gyulassy, H.-T. Elze, Ann. Phys. 173 (1987) 462.
[22] H.-T. Elze, M. Gyulassy, D. Vasak, Nucl. Phys. B 276 (1986) 706.
[23] D. Satow, H.-U. Yee, Phys. Rev. D 90 (2014) 014027, arXiv:1406.1150.
[24] K. Jensen, R. Loganayagam, A. Yarom, J. High Energy Phys. 1302 (2013) 088, arXiv:1207.5824.
[25] D.-F. Hou, H. Liu, H.-c. Ren, Phys. Rev. D 86 (2012) 121703, arXiv:1210.0969.
[26] T. Kalaydzhyan, Phys. Rev. D 89 (2014) 105012, arXiv:1403.1256.